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VISUAL WAVE OBSERVATIONS

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ABSTRACT

Visual wave observations will always be needed to supplement instrumental records. Ordinary visual observations are not adequate for estimation of the spectral components of the short-crested surface waves, but by the use of more precise visual observations utilizing all the waves passing a point, it is possible to determine the wave characteristics.

The most important observed wave characteristic is wave height. The theory of a wave record is ... the relationship
between ... spectrum. It is
shown th... be derived both
for the o... tributions using
only wav... es of recorded
visual ob... minimum of 50
to 100 w... ave height can
be estim...

Visual ob... more difficult
to interpr... ve distribution
must be ... e length" can
be estima... be estimated
from visua...

FOREWORD

The Hydrographic Office recently published a new wave forecasting manual, H. O. Pub. No. 603, prepared by Professor Pierson and his colleagues at New York University under contract with Project AROWA. To supplement this manual the Hydrographic Office herewith presents "Visual Wave Observations," also by Professor Pierson, as an explanation of the methods for obtaining wave observations in a manner compatible with the spectral forecasting technique.

Wave records are of two major types. Those obtained from wave staffs, pressure recorders, and other mechanical devices are accurate and reproducible, but they are also expensive and limited in number. Visual wave observations are subject to error, but they are readily obtained from shipboard as often as desired. In spite of the errors inherent in subjective estimates of wave characteristics, several important types of data can be secured from visual wave observations which can aid in wave forecasting. This report outlines the theory underlying visual wave observations and indicates the data that can be secured from them.

The Hydrographic Office is actively engaged in the development and operational testing of methods of wave forecasting. In order to increase the usefulness of the operational wave forecasts being issued by this Office, it is necessary to obtain more frequent and accurate synoptic wave reports. It is hoped that this report, which indicates how improved visual observations can be obtained, will encourage observers aboard ships to make observations in the method outlined.

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Captain U. S. Navy
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I. INTRODUCTION

Sea waves are irregular, confused, and sometimes mountainous. They are short-crested in that a given crest can be followed by eye along the crest for at most a distance of three or four times the distance between successive crests. Even the apparently more regular swell is still irregular in that there are considerable lengths of time during which the swell is very low in height. Swell is also short-crested, although swell crests are longer than sea wave crests. The visual observation of such an irregular pattern is a difficult and complicated procedure which needs to be described and interpreted precisely.

The main purpose of this paper is to describe wave irregularity and to present techniques for the visual observation of wave heights, "periods," "wave lengths," and "speeds." These techniques will make the values obtained by observation more useful because they will be more precise. A second purpose of this paper is to give the theoretical justification for the procedures given in the wave forecasting manual prepared by Pierson, Neumann, and James (1955).

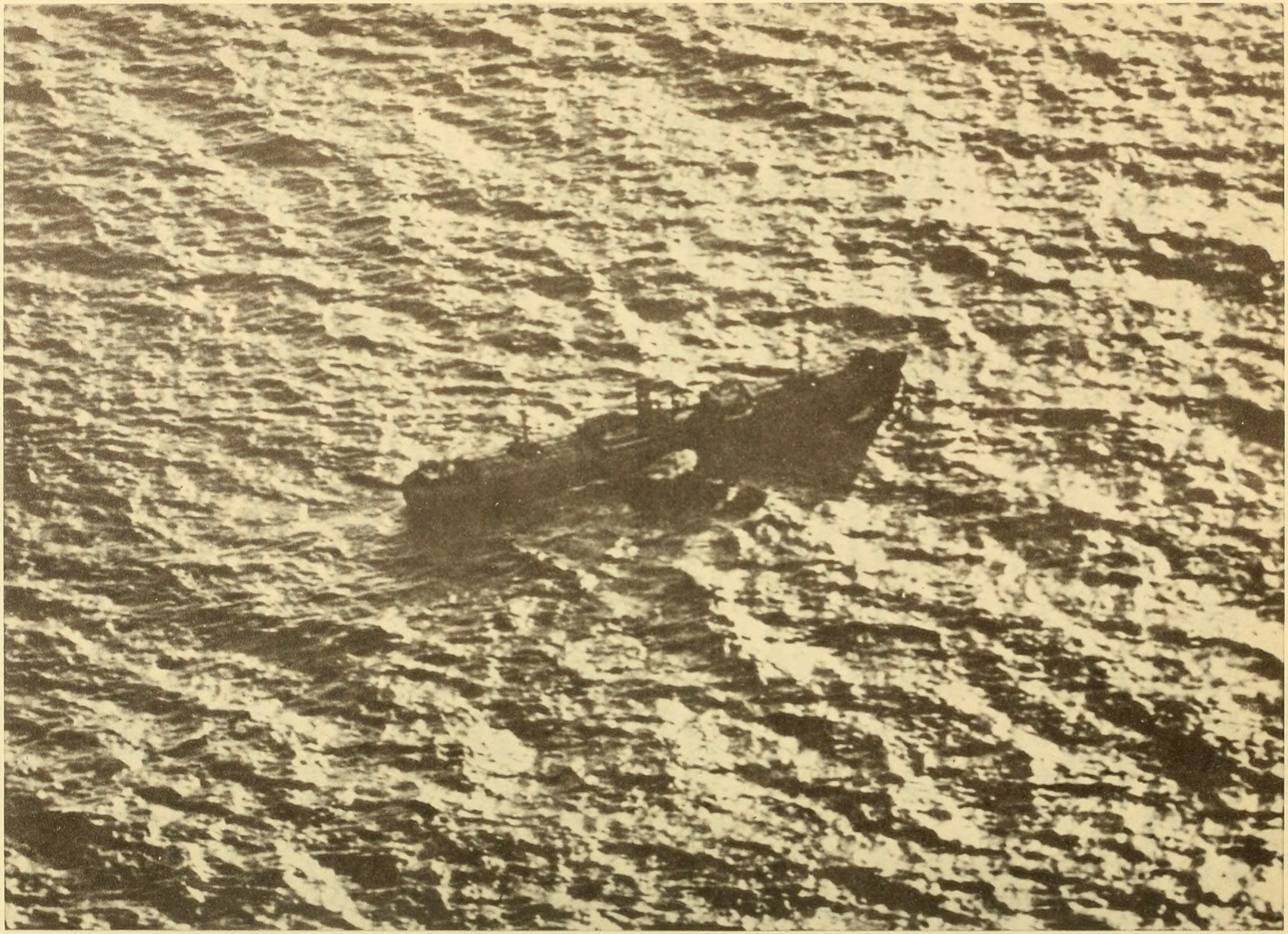
Visual height observations will never be as precise as instrumental observations of adequate duration as made by the various wave-pole techniques which have been developed for both deep and shallow water. The data obtained by the Hydrographic Office with instruments developed by the Beach Erosion Board will yield information which could never be obtained visually. However, visual observations will always be needed to supplement instrumental observations.

II. THEORETICAL AND PRACTICAL ASPECTS OF WAVE HEIGHT OBSERVATIONS

A. Techniques for Wave Height Observation

The mathematical representation of ocean waves as a short-crested Gaussian sea surface, as proposed by Pierson (1952, 1954), appears to represent the sea surface and its mathematical properties in a realistic way. This paper will be based on the results of this theory, but the practical aspects of the problem will be emphasized instead of the theoretical aspects.

Consider figure 1. It shows the irregular pattern of the sea surface as observed from an aircraft flying at a height of about 4,000 feet. One way to observe the wave heights would be to estimate or measure the crest-to-trough heights of the highest part of each short-crested



Photograph Courtesy of Capt. D. B. MacDiarmid, USCG

Figure 1. Aerial Photograph of Sea Surface.

wave by stereophotography. That is, a dominant wave could be selected in the stereo pair, and it could be followed along the crest until its highest part was found. The crest-to-trough height would then be found at this point. A large number of such observations could be made. The average of these values would then give some sort of average height.

Similarly, when a wave observer looks out over the sea surface, he tends to look at the highest part of each of the short-crested waves within his field of view. His eye skips about over the sea surface, and thus the values recorded are similar to the values described above.

The theory of the distribution of the values obtained in the observation of the highest part of each of the short-crested waves in the field of view of the observer has not yet been solved. The theoretical probability distribution of such heights is unknown, and it appears that it will remain unknown until some fundamental problems in time series are solved.

If the theoretical properties of an observed set of values are unknown, the observations are for all intents and purposes useless. To report that the average height of the waves as observed by this technique is so many feet does not permit an estimate of the higher waves or of their frequency of occurrence.

Most observations of wave heights at sea do not even possess the property of being the average of the highest part of each short-crested wave in the field of view of the observer. They are even cruder estimates of the "significant" height as made by looking out over the sea surface and guessing as to a characteristic height of the waves. Such estimates are unreliable because they depend subjectively on the observer and on the type of ship from which the observations are being made since the scale of the waves relative to the size of the ship influences the observer's choice of the characteristic height.

From the above discussion, it would appear that either just looking at the waves and assigning a characteristic height or just writing down a few heights of waves scattered about over the sea surface and computing the average is not an adequate method of visual observation. Knowledge of the wave height distribution, of the errors inherent in the sample size, and of source of observer error must be developed theoretically in order to make the interpretation of observed wave heights reliable.

Consider the observation of the heights of all waves that pass a fixed point. Such an observation could be made instrumentally by a wave-pole recorder, or it could be made just as easily by an observer if he knew that this was the correct procedure. Some of the theoretical properties of such a series of observations are known, and therefore their accuracy can be determined. The theory which is to be given is therefore based on the observation of the heights of all waves which pass a fixed point of observation. Once the distribution of the heights of all waves is known, it then becomes possible to omit the observation of some waves in a precisely defined way and still obtain reliable results.

The theory can be extended to cover the properties of the heights of all waves that pass (or are passed) by a moving point. Thus a point fixed in azimuth and distance relative to a moving ship can be used just as well as a stationary point.

The heights of the waves that pass a fixed point are lower than the heights of the highest part of each short-crested wave since the side of a short-crested wave can pass the point and the highest part can pass at a distance from the point of observation. Since these heights are the same as would be encountered by a ship under way, they are also of practical importance.

B. The Theory of A Wave Record

An ocean wave record is a sample from a quasi-stationary Gaussian process which is completely described by its energy spectrum. Much is known about the theory of such Gaussian processes since they have been studied extensively in electronics and in communication theory by Rice (1944), Wiener (1949), and Tukey (1949).

1. The Envelope

There are a number of ways to define the envelope of a wave record. For one way that is used, it can be shown that the envelope will touch each crest of the wave record only if the wave spectrum is narrow, and that the envelope is always distributed according to equation (1) as discussed below. For another way that is used, the envelope is defined to touch each horizontal part of the record, but then the probability distribution of the envelope reflects ripples and other minor (for this application) irregularities and only reduces to equation (1) for narrow spectra.*

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*Personal communication, R. A. Wooding; see also "Wind Generated Gravity Waves" by W. J. Pierson, Jr. (1954).

2. The Amplitudes

If the spectrum of the waves is narrow, the probability distribution of the amplitudes is known (Rice, 1944). As in figure 2, a sufficient number of amplitudes read from the record will have a known probability distribution function. If the spectrum is wide, the distribution is unknown. However, it would appéar from the theoretical results of Neumann (1953) that even a fully developed sea wave record will be approximately distributed in amplitudes according to this known distribution.

Given, then a wave record and a set of wave amplitude observations which are from a long enough record, the amplitudes will be distributed according to the law given by equation (1)

$$g(x)\,dx = \frac{2x}{E} e^{-x^2/E}\,dx \tag{1}$$

for $0 < x < \infty$,

which means that the probability that a given amplitude, say, ξ_n, will have a value between x and x + dx is given by equation (1). This probability distribution is often called the target distribution and also the Rayleigh distribution. It is related strongly to the Chi-square distribution.

The mean wave amplitude is found from equation (1) as in equation (2)

$$\int_0^{\infty} \frac{2x^2}{E} e^{-x^2/E}\,dx = -xe^{-x^2/E}\Big|_0^{\infty} + \int_0^{\infty} e^{-x^2/E}\,dx = \frac{\sqrt{\pi}}{2}\sqrt{E} = 0.886\sqrt{E} \tag{2}$$

The number E has the dimensions of square feet, and it represents the sum of the squares of all of the amplitudes of the infinite number of infinitesimally high sine waves which add together to make up the total wave record. The average amplitude of all the waves is equal to $0.886\sqrt{E}$.

The second moment about the origin of equation (1) can also be found. It is given by equation (3) since the integral from zero to infinity of equation (1) is equal to one.

$$\int_0^{\infty} \frac{2x^3}{E} e^{-x^2/E} dx = -x^2 e^{-x^2/E} \Big|_0^{\infty} + E \int_0^{\infty} \frac{2x}{E} e^{-x^2/E} dx = E \tag{3}$$

Another useful function derivable from equation (1) is the cumulative distribution function, which gives the probability that an amplitude will be less than or equal to the value x. It is given by equation (4).

$$F(x) = \int_0^{x} \frac{2\eta}{E} e^{-\eta^2/E} d\eta = 1 - e^{-x^2/E} \tag{4}$$

From this equation, table 1 or any other percentile distribution can be obtained.

Table 1 stops at 90 percent. One hundred percent of the waves have amplitudes less than infinity, which is all that can be said from equation (1) or equation (4). Theoretically, at least, a wave of very great amplitude is always possible.

Table 1

Wave amplitude data in cumulative ascending 10% values

10% less than	$0.32\sqrt{E}$
20% less than	$0.47\sqrt{E}$
30% less than	$0.60\sqrt{E}$
40% less than	$0.71\sqrt{E}$
50% less than	$0.83\sqrt{E}$
60% less than	$0.96\sqrt{E}$
70% less than	$1.10\sqrt{E}$
80% less than	$1.27\sqrt{E}$
90% less than	$1.52\sqrt{E}$

However, some very ingenious results of Longuet-Higgins (1952) can be used to avoid this difficulty. Longuet-Higgins studied the probability distribution of the highest wave amplitude out of N waves. From this he calculated the average value of the highest wave out of N waves. If a wave record containing a total of NM wave amplitudes is broken up into M pieces of N waves each, and if the M highest

amplitudes, one for each piece, are selected and averaged, the amplitude value is given by table 2. For a given observation of N amplitudes, the expected value of the highest amplitude is given by table 2. The very high waves are rare.

Table 2

Greatest wave amplitude data

N	Expected value of highest wave amplitude
20	$1.87\sqrt{E}$
50	$2.12\sqrt{E}$
100	$2.28\sqrt{E}$
200	$2.43\sqrt{E}$
500	$2.60\sqrt{E}$
1000	$2.73\sqrt{E}$

From the probability distribution function of the highest wave of N waves as given by Longuet-Higgins (1952), it is also possible to compute the most probable height, the height exceeded by 95% of the individual observations, and the height exceeded by 5% of the individual observations. These data are given in Pierson, Neumann, and James (1955).

The average amplitude of the K percent highest waves, as shown by Longuet-Higgins (1952), can be found from the appropriate truncated modification of equation (1). The value of X, say X_K, such that K percent of the waves have amplitudes greater than that value, is first found by solving (4) as given by equation (5).

$$1 - e^{-x_K^2/E} = 1 - (K/100)$$

or

$$e^{-x_K^2/E} = K/100 \tag{5}$$

The result is that the probability distribution function of the K percent highest wave amplitudes is given by equation (6).

$$g(x)\,dx = \frac{100}{K}\,\frac{2x}{E}\,e^{-x^2/E}\,dx \tag{6}$$

(for $X_K < x < \infty$ and zero otherwise)

In equation (6), the value of the integral is equal to one by virtue of the fact that X_K is chosen by (5). Stated another way, the probability distribution function* of the K percent highest waves is found from equation (1) by finding that part of the area to the right of a given point on the x axis such that it equals K percent of the total area, and then the truncated part is amplified 100/K times so that the area under the curve will again equal one.

The average amplitude of the one-tenth highest amplitudes is given by $1.800\sqrt{E}$, and the average amplitude of the one-third highest amplitudes is given by $1.416\sqrt{E}$.

3. Wave Heights

In a simple sine wave, doubling the amplitude gives the crest-to-trough height of the wave. In an irregular wave record this is not necessarily the case. A study of any wave record, as for example, figure 2, shows that a succeeding trough does not necessarily go as much below sea level as the crest it follows goes above sea level. In a swell (or equivalently, with a narrow spectrum), the succeeding trough is well correlated with the crest and the wave heights are approximately twice the crest amplitude. In a sea, where the spectrum is broader, this is not the case.

The results given above for wave amplitudes then need not give results applicable to crest-to-trough wave heights. Apparently, however, they apply, in many cases, to wave heights as well.

The results of Seiwell (1948) and Weigel (1949) in an analysis of wave heights bear this out in that such values as the ratio of the significant wave height to the mean wave height and the mean wave height to the average of the one-tenth highest waves all agree well with the theoretical values which would be obtained by doubling the values given above and interpreting the results as wave heights.

The most complete study of the problem is found in a paper by Watters (1953) where the crest-to-trough heights of 109 records were studied. It was found that the heights were distributed according to the distribution given by equation (1). The histograms given of the wave height distributions are just what would be expected from the sample size and the theory of sampling. The Chi-square test was applied to 38 of the records studied, and remarkably consistent results were obtained which conclusively prove that the distribution given by equation (1) is valid for the wave heights studied.

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*Hereafter this expression will be abbreviated to p.d.f.

The results cited unfortunately are not interpreted in terms of the spectrum of the waves. The records were pressure records in which the spectrum is made narrower by the filtering effect of depth; also, many may have been swell records with narrow band.

It may be that records of sea waves with a broad spectrum will have amplitudes distributed according to (1) and that the heights will not be distributed according to (1). On the other hand, the sharper peaks and shallower troughs which are the result of nonlinear effects may make the heights more like equation (1) and the amplitudes skewed in distribution.

When the wave amplitudes vary erratically from crest to succeeding trough, as in a sea, the theoretical distribution of the heights is a very difficult problem in probability theory because the succeeding trough is partially correlated with the crest.* If it were uncorrelated, the p.d.f. of the sum of the two values from equation (1) could be found and this would be the p.d.f. of the wave heights. This is not the case, however, and the true height distribution in a sea may depend in a complicated way on the spectrum of the waves. From these results several conclusions can be reached.

One conclusion, then, may well be that wave amplitudes follow equation (1), but that wave heights do not follow equation (1) in some circumstances. Another is that wave heights do follow equation (1) in some circumstances. The third is that the assumption that wave heights do follow equation (1) is about the best theoretical assumption that can be made at the present time because in many cases the assumption will be approximately correct, not leading to appreciable error, and because the complete theory has not been solved.

4. Spectrum Really Needed

If an actual ocean wave record is available, its amplitude distribution, or its height distribution, is not the most important property of the record. The energy spectrum of the record is what is really needed for many practical applications and without it little of real theoretical value can be accomplished.

C. Visual Height Observations

1. Reason Needed

There are many cases in which visual observations of ocean wave

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*As pointed out by Dr. Robin A. Wooding of New Zealand in personal correspondence.

heights are the only way to obtain wave data. They are also, in many cases, the only way to verify a wave forecast. A visual observation can never yield more than a crude estimate of the wave spectrum. However, it can be taken in a way which will yield an estimate of the average wave height, a verification of the distribution given by equation (1), and an estimate of the reliability of the observed values.

The results to be described are applicable to the type of observation made on ships at sea such as the sea state conditions reported by Atlantic Weather Patrol personnel and seamen on naval vessels. If these results are applied to such observations, the utility of the observations can be expected to increase greatly. The procedures to be described have been tested by Atlantic Weather Patrol personnel, and they do work.

2. Applicability of Theory

The theory discussed above has established the following points.

1. The waves must be observed at a fixed point or at a moving point because scatter-shot observations cannot be treated theoretically.

2. The p.d.f. of the amplitudes is very nearly given by equation (1) and the p.d.f. of the heights may well be approximated by doubling the values given in equation (1) and in the tables.

3. The successive waves are not independent occurrences and there is a correlation between successive amplitude values (and successive height values).

From these results, an attempt will be made to determine the reliability of wave height observations.

The wave observer is assumed to be keeping his eye on a fixed point on the sea surface (or on a point fixed relative to the moving vessel on which he is stationed). He sees a series of waves pass this point. He does not know the level of the sea surface and therefore he must estimate crest-to-trough wave heights. Waves with heights ranging from near zero to very large values will be passing the point at which the observers attention is fixed. It is assumed that the observer is estimating the heights as carefully as possible and that a tabulation of the observed heights is made.

A wave height is defined to be the difference between the height of the highest part of a mound above sea level and the deepest part of the

neighboring trench below sea level. Perturbations and smaller waves such as the one at $\widetilde{T}3$ in figure 5 are not counted if they do not pass through sea level.

The first danger which will make the observation less useful is a tendency to ignore the low waves. The observer in tabulating the heights of the passing waves may see many low waves and, because they appear insignificant compared to the more dominant waves, may fail to write them down. The average of the recorded values is then greater than the average of all the values, and since the nature of the omitted waves in unknown, the two values cannot be related.

The observer must therefore attempt to record the heights of all the waves that pass the point of observation if the observed values are to approximate the distribution given by equation (1).

This can be done, but it is difficult to do because almost invariably the observer will omit the low waves from such tabulations. Emphasis on this point and its importance may, in time, make it possible to obtain a complete sample of wave heights. If not, the theory of a truncated distribution can be used to avoid this difficulty in a way which will be discussed later.

3. The Average Height

After a sufficient number of heights has been recorded, the average height is the simplest and most useful statistic which can be computed from the data.

Even in a sea, if the heights are a correct estimate of E, the amplitudes are distributed according to equation (1).

Consider figure 2. The first wave height equals $\xi_1 + \xi_2$. The second wave height equals $\xi_3 + \xi_4$ and so on, therefore the average wave height is given by equation (7).

$$\overline{H} = \frac{(\xi_1 + \xi_2) + (\xi_3 + \xi_4) + \cdots + (\xi_{2N-1} + \xi_{2N})}{N} = \frac{2}{2} \sum_{i=1}^{2N} \frac{\xi_i}{N} = 2\,\overline{\xi} \qquad (7)$$

Even if the successive amplitudes combine to give height values unrelated to the target distribution, the average wave height is still equal to twice the average wave amplitude, and, under the assumption that the amplitudes are from a target distribution, the value of E can be estimated.

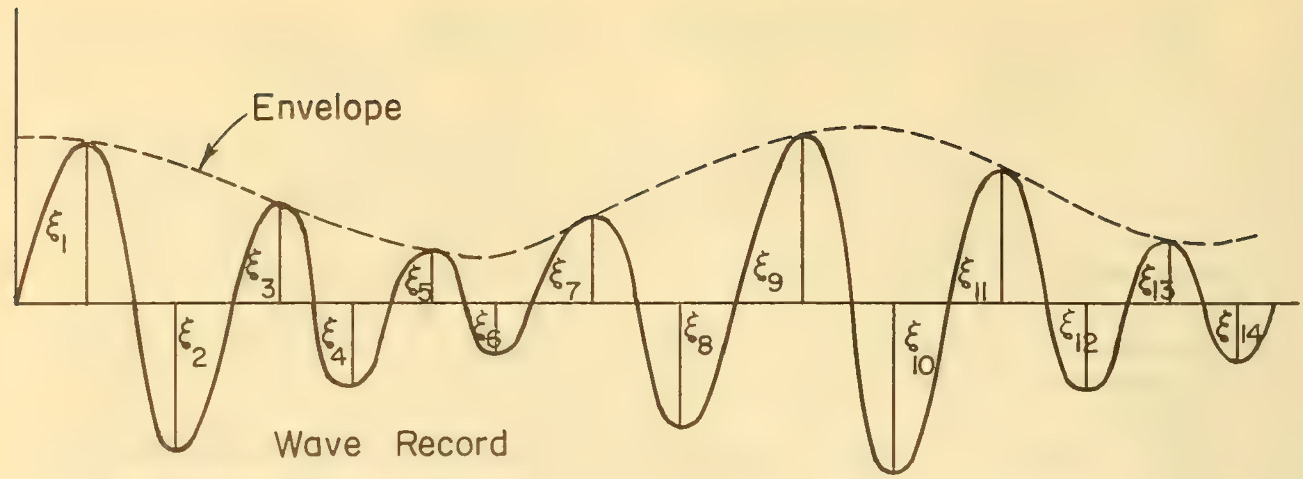

Figure 2. Envelope and Amplitudes of a Wave Record.

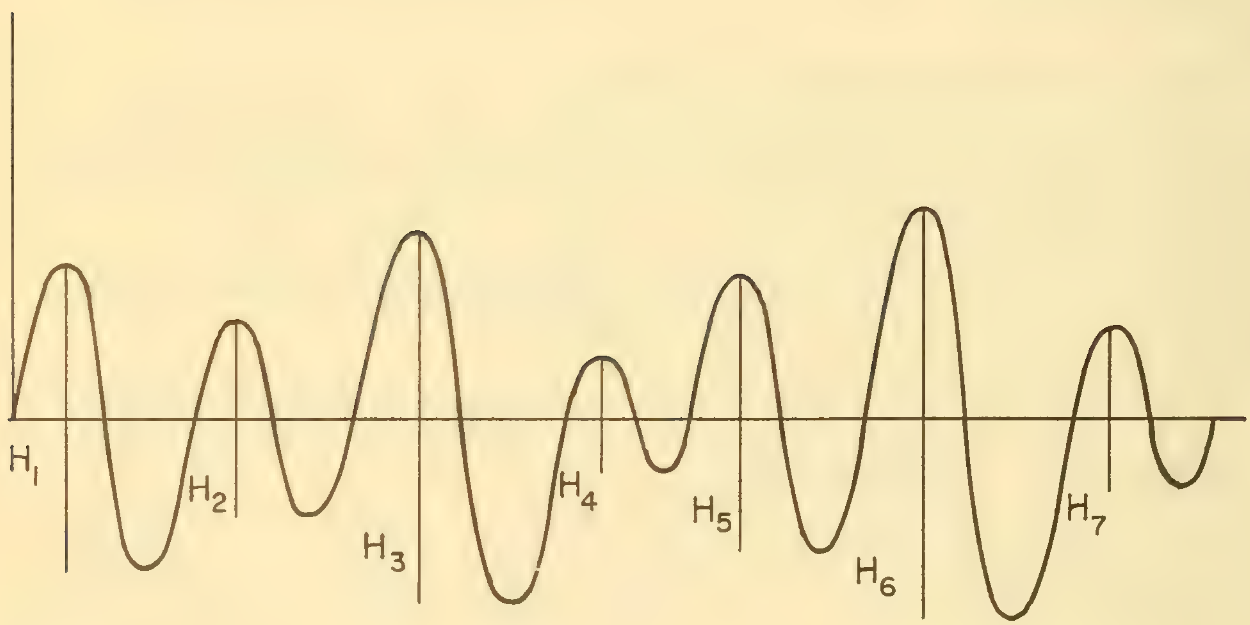

Figure 3. An Artificial Wave Record.
Trough Amplitude Equals Preceding Crest Amplitude.
Heights Are Uncorrelated.

4. Reliability of the Average Height

The expected value of the average height equals the expected value of a random variable from the population. Therefore the average value of a sample of N observations is an unbiased estimate of the mean of the theoretical population. There will be no systematic error in the estimate of the average.

If enough assumptions are made about the nature of the observations, the confidence limits of a particular estimate of an average wave height can be found. These assumptions are not too realistic. They will have to be modified qualitatively after the derivation is complete, but at least they permit the statement of some practical rules applicable to height observations. The following assumptions are made:

1. Each trough is correlated with the value of unity to the preceding crest.

2. The crests are completely uncorrelated (i.e., the height of a wave is independent of the height of preceding and following waves).

3. The amplitudes, and therefore the heights, by reason of the first assumption are distributed according to the target distribution.

A total of N wave heights then corresponds to a total of N independent amplitude observations, and a wave record with these properties (it does not exist) would look like the record sketched in figure 3. (It is interesting to note that the amplitude values could be completely uncorrelated and that the autocorrelation function of such a record would still show a well-developed oscillation through plus and minus values.)

A theorem in statistics can now be used to study the average value of these N wave amplitudes. The central limit theorem of statistics states (Cramer, 1946, page 215, for example) that: "If ξ_1, ξ_2..... ξ_n are independent random variables all having the same probability distribution, and if m_1 and σ_1 denote the mean and the standard deviation of every ξ_ν, then the sum

$$\xi = \sum_1^N \xi_\nu \tag{8}$$

is asymptotically normal".....with a mean Nm_1 and the standard deviation,

$\sigma_1 . \sqrt{N}$ "It follows that the arithmetic mean

$$\bar{\xi} = \frac{1}{N}\sum_{1}^{N} \xi_{\nu} \tag{9}$$

is asymptotically normal........" with a mean m_1 and a standard deviation, $\sigma_1 / \sqrt{N}$."

This central limit theorem works remarkably well in many cases for small values of N. For example, the faltung (or convolution) of three samples from a rectangular distribution even then approximates a normal distribution, and for the target distribution, it may well be that the theorem is applicable for values of N as low as 9 or 16 for practical purposes.

It is necessary to calculate the second moment about the mean of the distribution given by equation (1). This is found by means of equation (10).

$$\mu_2 = \sigma_1^2 = \int_0^{\infty} \left(X - \frac{\sqrt{\pi}\sqrt{E}}{2}\right)^2 \frac{2X}{E} e^{-X^2/E}\, dX$$

$$= \int_0^{\infty} \frac{2X^3}{E} e^{-X^2/E}\, dx - \frac{2\sqrt{\pi}\sqrt{E}}{2} \int_0^{\infty} 2x^2 e^{-X^2/E}\, dx + \frac{\pi}{4} E \int_0^{\infty} \frac{2X}{E} e^{-X^2/E}\, dx \tag{10}$$

$$= E - \frac{2\pi}{4} E + \frac{\pi}{4} E$$

$$= E\left(1 - \frac{\pi}{4}\right)$$

From the above theorem, the computed mean of a sample of N observed wave amplitudes will be normally distributed with a mean given by equation (2) and a second moment about the mean given by equation (10) divided by N.

The variable z defined below is therefore distributed according to a normal distribution with a zero mean and a unit standard deviation.

$$Z = \frac{\frac{1}{N}\sum_{1}^{N} \xi_{\nu} - \frac{\sqrt{\pi}}{2}\sqrt{E}}{\sqrt{\left(1 - \frac{\pi}{4}\right) E / N}} \tag{11}$$

It is now possible to compute the probability that z will lie between any two values under the unit normal curve. The probability that z will

lie between the two values -B and +B as given by the area under the curve from -B to +B can be computed and set equal to A as in equation (12).

$$P(-B < Z < B) = A \tag{12}$$

If, for example, B equals 1.65, then A is 0.90. That is, nine times out of ten, z will lie between -1.65 and +1.65.

Some operations on equations (11) and (12) can now be carried out. The first result is equation (13), and the purpose of the operations is to get the symbol $\sqrt{E}$ inside the inequalities and $\bar{\xi}$ outside the inequalities.

$$-B < \frac{\bar{\xi} - (\sqrt{\pi}/2)\sqrt{E}}{\sqrt{(1-\frac{\pi}{4})E/N}} < B. \tag{13}$$

Equation (13) can be written as

$$-B < \frac{\bar{\xi}}{\sqrt{(1-\frac{\pi}{4})E/N}} - \frac{\sqrt{\pi}}{2\sqrt{(1-\frac{\pi}{4})/N}} < B. \tag{14}$$

This yields

$$-B + \frac{\sqrt{\pi}}{2\sqrt{(1-\frac{\pi}{4})/N}} < \frac{\bar{\xi}}{\sqrt{(1-\frac{\pi}{4})E/N}} < B + \frac{\sqrt{\pi}}{2\sqrt{(1-\frac{\pi}{4})/N}} \tag{15}$$

and

$$\frac{-B\sqrt{(1-\frac{\pi}{4})/N} + \sqrt{\pi}/2}{\bar{\xi}} < \frac{1}{\sqrt{E}} < \frac{B\sqrt{(1-\frac{\pi}{4})/N} + \sqrt{\pi}/2}{\bar{\xi}} \tag{16}$$

Finally, by inverting equation (16) the result is

$$\frac{\bar{\xi}}{\sqrt{\pi}/2 + B\sqrt{(1-\frac{\pi}{4})/N}} < \sqrt{E} < \frac{\bar{\xi}}{\sqrt{\pi}/2 - B\sqrt{(1-\frac{\pi}{4})/N}}. \tag{17}$$

The mean amplitude gives an estimate of E which will be called E_m

as defined by equation (18).

$$\bar{\xi} = \frac{\sqrt{\pi}}{2} \sqrt{E_m} \tag{18}$$

When this is substituted into equation (17) and when simplifications are made, the final result is given by equation (19).

$$\frac{\sqrt{E_m}}{1 + \frac{B}{\sqrt{N}} \sqrt{\frac{4}{\pi} - 1}} < \sqrt{E} < \frac{\sqrt{E_m}}{1 - \frac{B}{\sqrt{N}} \sqrt{\frac{4}{\pi} - 1}} \tag{19}$$

The value of $\sqrt{E_m}$ is obtained from the observation of the waves. Usually it is not equal to the true value of $\sqrt{E}$. From the above inequality the bounds within which the true value of $\sqrt{E}$ will lie, say, 90% of the time, can be found by multiplying $\sqrt{E_m}$ by the factors determined from (19) with B equal to 1.65 and N equal to the number of observations. For typical values of N, the results of equation (19) yield the values given in table 3 as entered in the columns marked "theoretical."

Table 3

Confidence values of $\sqrt{E_m}$

N	Lower Value (Safety factor)	Lower Value (Theoretical)	Upper Value (Theoretical)	Upper Value (Safety factor)
9	$0.71 \sqrt{E_m}$	$0.78 \sqrt{E_m}$	$1.40 \sqrt{E_m}$	$1.68 \sqrt{E_m}$
16	$0.76 \sqrt{E_m}$	$0.82 \sqrt{E_m}$	$1.28 \sqrt{E_m}$	$1.44 \sqrt{E_m}$
25	$0.80 \sqrt{E_m}$	$0.85 \sqrt{E_m}$	$1.21 \sqrt{E_m}$	$1.33 \sqrt{E_m}$
50	$0.85 \sqrt{E_m}$	$0.89 \sqrt{E_m}$	$1.12 \sqrt{E_m}$	$1.21 \sqrt{E_m}$
100	$0.89 \sqrt{E_m}$	$0.92 \sqrt{E_m}$	$1.09 \sqrt{E_m}$	$1.12 \sqrt{E_m}$
200	$0.92 \sqrt{E_m}$	$0.94 \sqrt{E_m}$	$1.06 \sqrt{E_m}$	$1.09 \sqrt{E_m}$

The techniques used in the above derivation are a standard part of statistical theory. For another example of how they may be applied, see Wilks (1951), pp. 195 through 201.

Exactly the same factors multiply the average height values or the significant height values, since both are simply constants times $\sqrt{E}_m$.

For example, if 25 values of ξ were obtained from the artificial wave record given in figure 3 under the assumptions which were listed, and if the observed mean wave amplitude was 10 feet, then the true wave amplitude as computed from a much larger sample from the same population would be between 8.5 feet and 12.1 feet for 90% of such experiments. If on the other hand the value was based on 100 amplitudes, the true value would be between 9.2 feet and 10.9 feet for 90% of such experiments.

What has the above derivation to do with actual ocean waves since the properties assumed in the derivation were shown not to be properties of actual waves? The answer is that it appears that the values given are the narrowest bounds possible and that the effect of correlation is always to make the bounds even wider. That is, if the above theory says that the bounds are between, say, 7 feet and 13 feet for a given estimate, then the effects of correlation in the heights make the true bounds even greater, say, from 5 feet to 15 feet.

The true confidence limits of such an estimate will probably not be known until the study of time series has advanced in this theoretical direction. The range of the theoretical bounds as given above is an underestimate of the range of the true bounds. The true upper bound is greater than the theoretical upper bound, and the true lower bound is less than the theoretical lower bound.

The model that was made up applies fairly realistically to a series of observations of wave heights as would be made in an actual visual observation. The correlation of unity between a crest amplitude and a succeeding trough amplitude makes each wave height an independent observation instead of the sum of two independent observations which is realistic in the sense that wave heights have been observed to be distributed according to equation (1). Thus the average of N wave heights should be considered to be the average of N independent observations, and table 3 would apply to the computed values.

The assumption of independence for the individual height values is more to be questioned. As stated above, the effect of correlation is

to spread out the confidence limits, since the effective number of observations is decreased by the correlation between values. Another model could be constructed in which two successive complete waves are correlated with the value unity and all parts are independent. Such a record (again an impossible one) would be like figure 4.

Then N wave height observations are really N/2 independent observations. With the use of N/2 instead of N in table 3, the confidence limits given in table 3 with a safety factor were obtained. This safety factor is thus a very crude attempt to estimate the effect of stronger correlation between successive waves.

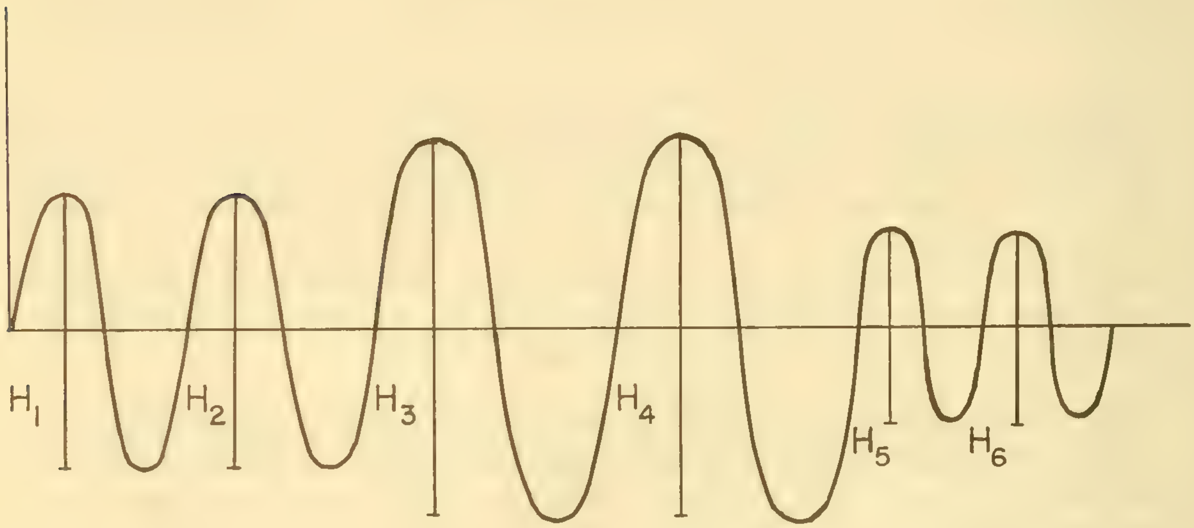

Figure 4. An Artificial Wave Record.
Trough Amplitude Equals Preceding Crest Amplitude.
Heights Correlated with the Value of Unity, Two by Two

5. Practical Conclusions

From these results, it is evident that at least 50 wave heights must be observed and tabulated before a reliable estimate of the average wave height (or $\sqrt{E}$) can be found. Very little confidence can be placed in an estimate based on 25 values; 100 values would be much more reliable. If 100 values are observed, if every wave

is recorded, and if the average wave "period" is 10 seconds, then such an observation requires nearly 20 minutes to make. A wave observer must take sufficient time to make an adequate observation. A good observation simply cannot be made in just a few minutes.

6. Truncated Distribution

As the wave observer looks at the waves, there are times when low waves pass. For example, with E equal to 100 ft.^{2}, the significant height would be 28.3 feet. One wave out of 10 would be less than 6.4 feet high; compared with the more dominant waves, there would be a very strong tendency to ignore some of these low waves.

When this is done, the problem is what to do with the observed values which have now become a sample from an unknown population, since the probability that the observer will ignore a given low wave is an unknown factor.

The first thing that must be done is to truncate the theoretical distribution sharply. That is, equation (1) must be set equal to zero for all x less than a certain value and then correctly normalized. If the low waves are to be ignored in the tabulation, then all low waves must be ignored and not just some unknown and unspecifiable fraction of the values within a certain class interval.

There are two ways to truncate the distribution. The first way is to discard a certain fixed percentage of the lowest waves of all the waves that pass a given point. The second way is to discard all waves less than a preassigned height value.

7. Truncated Distribution at a Fixed Percentage

The first way, namely discarding a certain fixed percentage of all the lowest waves to pass a fixed point of observation, is inherent in the concept of the significant height. The significant height is the average of the heights of the 33 percent highest waves to pass the point of observation.

To observe the significant height correctly, the following procedure could be used. The observer would watch the waves pass the fixed point. If a high wave passed, he would note down its height. If a low wave passed he would simply make a check to note the passage of the wave. A series of recorded heights and checks would be the result. The total number of heights and checks, say M, would be counted up, and the sum divided by 3. The observed heights would then be put in

descending order, and the highest M/3 values selected. The average of the M/3 highest waves would then be the significant height.

As an example, consider the series of recorded values given below as they may have occurred in an observation:

√,	√,	√,	6,	√	√,	6,	8,	√,	√
√,	6,	6,	8,	10	12,	8,	6,	6,	√
√,	6,	6,	8,	8	10,	12,	14,	16,	12
8,	6,	6,	√,	√	√,	√,	√,	6,	6
6,	6,	8,	8,	10	8,	6,	6,	√,	√
√,	√,	√,	6,	6	8,	6,	8,	√,	√
√,	√,	√,	√,	√	6,	6,	8,	8,	10
10,	10,	8,	6,	6	√,	√,	√,	6,	6
6,	6,	8,	8,	8	6,	8,	10,	8,	8
6,	√,	√,	√,	√	6,	6,	√,	√,	√

There is a total of 100 height observations. There are 37 check values for low waves less than 6 feet high. There are 31 six-foot waves, 20 eight-foot waves, and so on.

The lowest two-thirds of the waves must be eliminated from the computations, so the sixty-seven lowest waves must be left out. The thirty-seven lowest waves automatically drop out, and then thirty of the six-foot high waves are eliminated. Thus the one-third highest waves consist of the values of the heights greater than six feet and one six-foot wave to make up a total of 33.

The average of the one-third highest waves is then computed according to the following procedure:

Height	Number	Product
6	1	6
8	20	160
10	7	70
12	3	36
14	1	14
16	1	16
TOTAL	33	302

$$\text{Significant height} = \frac{302}{33} = 9.2 \text{ feet}$$

$$\sqrt{E} = \frac{9.2}{2.83} = 3.25.$$

There are a number of disadvantages to this procedure. The total number of waves which pass must still be counted. How can the one-third highest waves be counted if the two-thirds lowest waves are not counted also, so that it will be known that the one-third highest waves are actually some one-third of a total number of waves?

Also if the significant height is computed, a large number of the waves which pass cannot be utilized in computing a statistic about the height. Many fewer usable values are obtained during a given time duration for the observation. A lot of time is wasted doing nothing.

With the aid of the truncated distribution for K equal to 33 percent, the mean of the distribution and the standard deviation could be found. Then the steps used above to determine confidence limits for samples from the complete distribution could be used on the mean and second moment of the truncated distribution to determine the confidence limits of a significant height determined from N observed values. The results would be more reliable for a given N because some (but not all) of the correlation effect would be removed. However it would take three times as long to observe the N elements of the sample.

Exactly similar procedures could be used with any other percentage of the highest waves. However, in each case the total number of waves which pass must be observed.

8. Truncated Distribution at a Fixed Height

The second way to truncate the distribution is to eliminate all waves less than a certain fixed height and observe every wave in excess of this fixed height. For a given state of the sea the observer might record all heights greater than, say, 4 feet. For a higher sea all waves in excess of 10 feet could be recorded.

It is then possible to compute the average of the observed values and from this the true average of all the heights, including those which were not recorded, and any other desirable height parameter. The theoretical derivation is given in the following paragraphs.

Let the minimum height recorded be equal to $H_{min.}$. Then $\xi_{min.}$ equals $H_{min.}/2$, and the theory will be worked out using amplitudes. The results must be doubled at the finish to obtain the height parameters needed.

Since $$\int_{\xi_{min.}}^{\infty} \frac{2x}{E} e^{-x^2/E}\, dx = 1 - e^{-\xi^2_{min.}/E}, \qquad (20)$$

the truncated distribution is given by

$$g(x) = \frac{2xe^{-x^2/E}\,dx}{E(1-e^{-\xi^2_{min.}/E})} \qquad \text{for } \xi_{min.} < x < \infty. \tag{21}$$

The percent of the low waves omitted simply equals

$$100\,(1-e^{-\xi^2_{min.}/E}). \tag{22}$$

The average amplitude of all of the waves that are higher than $\xi_{min.}$ is given by the first moment of equation (21), and the evaluation of the integral yields the following result for $\overline{\xi}$ * which is defined to be the average amplitude of all waves in excess of $\xi_{min.}$

$$\overline{\xi}^* = \int_{\xi_{min.}}^{\infty} \frac{2x^2e^{-x^2/E}\,dx}{E(1-e^{-\xi^2_{min.}/E})} = \frac{-xe^{-x^2/E}}{1-e^{-\xi^2_{min.}/E}}\Bigg|_{\xi_{min.}}^{\infty} + \int_{\xi_{min.}}^{\infty} \frac{e^{-x^2/E}}{1-e^{-\xi^2_{min.}/E}}\,dx$$

$$= \frac{\xi_{min.}e^{-\xi^2_{min.}/E} + \int_{\xi_{min.}}^{\infty} e^{-x^2/E}\,dx}{\left[1-e^{-\xi^2_{min.}/E}\right]} \tag{23}$$

The last integral in equation (23) is the integral of the normal distribution between known limits. It can easily be evaluated from tables.

Equation (23) is a function of three variables, $\overline{\xi}$*, ξ min., and E. Any two determine the third. Suppose then that the heights of all waves greater than 4 feet are recorded, and that the significant height of the waves is 8 feet. The significant height determines E, and then $\overline{\xi}$ * can be computed. Under these conditions the average height of all waves greater than 4 feet is 6.64 feet. The percent of waves omitted can be found from (22) and in this case it equals 39.4 percent.

Table 4

The significant height, average height, and percent of waves omitted in terms of the average height of all waves in excess of 4, 10, or 20 feet

AVERAGE HEIGHT OF ALL WAVES GREATER THAN 4 FEET	4.86	5.30	5.70	6.16	6.64	7.22	7.68	8.80	9.82	11.04	12.3
AVERAGE HEIGHT OF ALL WAVES	2.50	3.13	3.75	4.37	5.01	5.63	6.25	7.50	8.76	10.0	11.3
SIGNIFICANT HEIGHT	4	5	6	7	8	9	10	12	14	16	18
% OF WAVES OMITTED	86.6	72.2	59.0	48.2	39.4	32.9	27.6	20.0	14.5	12.4	10.0
AVERAGE HEIGHT OF ALL WAVES GREATER THAN 10 FEET	12.06	12.88	13.50	14.7	15.7	16.6	18.52	21.08			
AVERAGE HEIGHT OF ALL WAVES	6.25	7.50	8.76	10.00	11.3	12.5	15.0	17.8			
SIGNIFICANT HEIGHT	10	12	14	16	18	20	24	28			
% OF WAVES OMITTED	86.5	75.1	63.9	54.2	46.3	39.4	29.5	23.2			
AVERAGE HEIGHT OF ALL WAVES GREATER THAN 10 FEET	22.7	25.4	27.5	30.4	33.4	36.8	39	43			
AVERAGE HEIGHT OF ALL WAVES	20.0	22.5	25.0	28.1	31.3	34.3	38.8	40.7			
SIGNIFICANT HEIGHT	32	36	40	45	50	55	60	65			
% OF WAVES OMITTED	18.0	14.5	12.4	10	7.5	6.5	5.6	4.6			
AVERAGE HEIGHT OF ALL WAVES GREATER THAN 20 FEET	33.16	35.3	38.3	42	42.8	47.2	48.4	54.8	63	66	
AVERAGE HEIGHT OF ALL WAVES	25	28.1	31.3	34.3	38.8	40.7	43.7	50.1	56.3	62.5	
SIGNIFICANT HEIGHT	40	45	50	55	60	65	70	80	90	100	
% OF WAVES OMITTED	39.4	32.9	27.6	23.7	20	17.5	14.5	11.5	10	7.5	

Table 4 gives the data needed for typical significant heights. In an actual observation the number obtained is the average height of all waves in excess of a certain height, and the table then gives the true significant height, the true average height of the total sample, and the percent of waves omitted.

In each entry, the average height of all waves is less than the average height of all waves in excess of some fixed height. Thus the tendency to ignore the low waves can make a visual observation quite unreliable unless corrections for the omitted waves are made theoretically. Note also that as the percent of waves omitted becomes smaller the difference between the average of the truncated distribution and the average of the full distribution becomes less and less and less.

9. An Example

The data obtained by the USCGC UNIMAK from 1800 to 1900Z on February 14, 1953 give an example of the procedures which can be employed in the use of the theory of a truncated distribution.

The original raw data were first of all averaged to determine the average height of the reported waves. Then the value of $\sqrt{E}$ was computed. From this and table 1 the theoretical distribution can be compared with the observed values. The result is given in table 5.

Table 5

Data obtained by USCGC UNIMAK 141800Z to 141900Z of February 1953

Average Height 15.5 feet (uncorrected)

Significant Height 24.5 feet (uncorrected)

Limits	Theoretical Frequency	Observed Frequency	Error
0 - 5.6	5	0	-5
5.6 - 8.2	5	9	+4
8.2 - 10.5	5	6	+1
10.5 - 12.4	5	5	0
12.4 - 14.5	5	0	-5
14.5 - 16.8	5	6	+1
16.8 - 19.3	5	7	+2
19.3 - 22.2	5	10	+5
22.2 - 26.6	5	6	+1
26.6	5	1	-4
TOTAL	50	50	

From the table it is seen that the waves between heights of zero and 5.6 were simply not observed although there should have been about five of them in a sample of fifty values under the assumption that the true mean was 15.5 feet. Also the mean of 15.5 feet implies that there should have been five waves greater than 26.6 feet and only one such wave was actually observed.

Now, the heights of all waves in the original sample which are greater than 10 feet can be averaged. The result is an average of 17.1 feet, and by the application of the results given above it follows that a better estimate of the significant height is 20.9 feet and that a better estimate of the average height of all waves is 13.1 feet. The results are summarized in table 6.

Table 6

Corrected data on the basis of the theory of truncated distributions

Average of heights greater than or equal to 10 ft. = 17.1 ft.
Significant height = 20.9 ft.
True average height = 13.1 ft.

Limits	Theoretical Frequency	Observed Frequency	Error
10	26	9	-17 (not effective)
10-12	11	9	-2
13-15	9.8	3	-6.8
16-18	8.5	10	+1.5
19-21	5.9	9	+3.1
22-24	3.3	5	+1.7
25-27	2.6	3	+0.4
27	(less than 0.5)	0	

With a lower true average height the predicted number of waves with large height values agrees much better with the observations. Or, stated another way, table 6 agrees with the theory much better at the high end of the distribution than table 5. Another important point to note is that of the twenty-six waves less than ten feet in height, which in all probability actually passed during the time of observation, only 9 were observed. The effect of this omission must be to increase the computed average of the uncorrected results to a value greater than the true average.

Table 7

Comparison of corrected and uncorrected visual height observations with forecast values, February 1953

	Date	Time	Position	Observed sig. ht. (no correction	Observed sig. ht. (corrected)	Decrease	Forecast sig. ht.	Error (uncorrected obs.)	Error (corrected obs.)	90% confidence limits on corrected values	Band error 50 obs.	90% safety factor	Band error 25 obs.
MENDOTA	2/9	12Z	51°43'N	34	28	6	20	-14	-8	25-32	-5	25-34	-4
		18Z	39°24'W	27	21	6	11	-16	-10	19-24	-8	18-25	-7
BARATARIA	2/10	12Z	36°40'N	20	17	3	22	+2	+5	15-19	+3	14-20	+2
		18Z		17	14	3	24	+7	+10	12-16	+8	12-17	+7
	2/11	12Z	69°35'W	15	13	2	11	-4	-2	11-15	0	11-16	0
		21Z		11	9	2	6	-5	-3	8-10	-2	8-11	-2
	2/16	12Z		32	27	5	24	-8	-3	24-31	0	23-33	0
		21Z		31	26	5	14	-17	-12	23-30	-9	22-31	-8
DUANE	2/10	18Z	35°00'N	21	18	3	13	-8	-5	16-20	-3	15-22	-2
	2/21	12Z	48°00'W	19	18	1	18	-1	0	16-20	0	15-22	0
UNIMAK	2/14	13Z	44°00'N	32	27	5	18	-14	-9	24-31	-6	23-33	-5
		18Z	41°00'W	25	21	4	18	-7	-3	19-24	-1	18-25	0
Sum obs. values				284	239	45	199	103	70		45		37
Sum with regard to sign								-85	-40		-23		-19
Average				23.7	19.9	3.75	16.1	8.6	5.8		3.75		3.1
Bias								-6.6	-3.3		-1.91		-1.58

10. An Overall Check of the Theory.

During a two-week period in February 1953, the Weather Bureau observers on the various coast guard cutters of the Atlantic Weather Patrol made visual wave observations which were to be used as a check of the forecasting procedures given by Pierson, Neumann, and James (1955). A check for internal consistency of the data showed that the observations had been carefully made.

The height observation for a particular report consisted of the tabulation of 50 individual crest-to-trough heights. A considerable range of height values was reported for each observation. For a given observation, the reported height values would range from 10 feet to 30 feet, which appeared to agree with the theory that the wave heights were distributed according to equation (1). The average of the 50 reported heights was computed, and the significant height was found from the average height by multiplying the average height by 1.416/0.886 (or 1.60), which is the ratio of the significant height to the average height.

Twelve of the reported observations were selected because of the high waves that were present. Forecasts based on the theory of the manual by Pierson, Neumann, and James (1955) were prepared without knowledge of the observed values. The results of the comparison of the forecasted values with the observed significant heights computed as described above are presented in table 7 under the heading, Observed Significant Height (no correction) and Forecast Significant Height.

The results of the comparison of the forecast and observed values were most disappointing. Errors as big as 17 feet resulted. The average forecast error was 8.6 feet. There was also a definite bias in that all but two of the forecast values were less than the observed values. The column labeled Error (uncorrected observations) shows these results. There was evidently something wrong!

At that time, none of the work on confidence limits or on truncated distributions as discussed above had been applied to the data although the theory was a standard part of statistical texts as it is given, for example, by Cramer (1946) and Wilks (1951). The theories discussed above were then developed and applied to the observational data.

A closer look at the reported height values shows that the heights did not follow the distribution given by doubling the coefficients in table 1. The low waves predicted by the probability distribution function were either missing or reported in far too small a proportion.

However, it was known that equation (1) was quite likely to be the true distribution of all wave heights as evidenced by the works cited above; and in addition the forecasting method which was being tested had worked well when compared with actual wave record observations.

It was evident, therefore, that the observers had not been able to observe and record the low waves that had actually occurred. As shown in the UNIMAK example, with a significant wave height of 20.9 feet, 26 waves out of 67 should have had a height less than 10 feet. Only 50 waves were actually recorded, and 17 waves less than 10 feet high were omitted.

Histograms of the data were plotted, and the observations were truncated at that height such that the distribution above that height resembled a truncated distribution as given by equation (21). From this truncated distribution the new corrected average height and corrected significant height were computed.

The values which resulted are entered in table 7 under the heading Observed Significant Height (corrected). The result was to decrease each value by an amount which depended upon the nature of the original sample. Some heights were decreased by as much as six feet, and the average decrease was 3.75 feet. The decrease is tabulated under the entry labeled, Decrease.

The forecast values and the corrected observed values then agreed far better than the forecast values and uncorrected observed values. Some of the largest errors were decreased a great deal. Seven of the twelve forecasts were within plus or minus five feet of the observed values. The forecast values still had a tendency to be lower than the observed values.

When the truncated distribution is used to determine the significant wave height, the confidence limits determined from the full distribution, strictly speaking, should not be used to obtain estimates of the reliability of the observations. The correct procedure would be to use the mean and second moment about the mean of the truncated distribution in a derivation similar to the one given above.

However, such a derivation would have to be carried out for many different cases, and it is believed that the final results would not improve too much on the estimates obtained from the theory derived above as based on the full distribution.

The confidence limits derived above can be applied to the results

obtained in table 7 with the reservation that the results are only approximate. In table 7, the 90 percent confidence limits are given on the assumption that the observations consisted of 50 independent height values. For example, for the observations made by the USCGC MENDOTA on 2 September 1953 at 1200Z, the Observed Significant Height (corrected) was 28 feet, and 90 percent of the time (under the assumptions which were made) the true significant height would be between 25 and 32 feet on the basis of many more observations.

The forecast error as a departure of the forecast value from the closest value of the 90 percent confidence limits is then entered as the band error for 50 independent observations.

The band error is a better measure of the discrepancy between the forecast and observed values because it does not penalize the forecast value for the unavoidable observational error which is due to the small sample size.

Three of the twelve forecasts are within the 90 percent confidence of the observations. Four more are within three feet of the 90 percent confidence limits. When the band error for an assumed 50 independent height observations is studied, it is seen that the forecasts are quite accurate.

The last two entries of the table show the 90 percent confidence limits on the assumption that the heights are really only 25 independent observations. This permits a spread of ten feet between the upper and lower bounds of some of the limits. For more precision it is evident that visual observations should consist of 100 observations at least, in order that it would be possible to be sure of somewhere near 50 independent values.

Under these conditions, four forecasts fall within the 90 percent confidence limits. Five more fall within five feet of the 90 percent confidence limits. Under these conditions, though, the confidence limits are so broad that the observations are of little use in saying anything about the wave properties. One of the purposes of this paper is to show that reliable observations are needed and that they cannot be reliable if enough individual values are not observed.

There is a consistent bias running through the data. The observed values consistently run higher than the forecast values. Much more data need to be collected before this bias can be established as real or false. There is, though, a possible explanation for this bias. It is that the observers did not keep an eye exactly at a fixed point on

the sea surface. If each observation had a little extra height added to it as the observer looked along the crest to the highest part of the crest, then the average of these heights would tend to be higher than the average of the heights of the waves passing a fixed point.

11. Other Errors

There is finally the question of the reliability of the height estimates as made by visual observations. Can an observer estimate the wave height of a wave thirty feet high within plus or minus two or three feet? Any such error, if consistent, in the estimation of the individual wave heights would introduce errors in the reported values. Very little is known about the nature of such errors, but there does seem to be a tendency to overestimate wave heights when visual observations are made. A cheap easily used instrumental aid for the measurement of wave heights would be a very useful device to be supplied to ship's personnel taking wave observations if such an instrument could be devised.

When the possibility of observer error, in addition to statistical error, is considered, it is seen that the results of table 7 are a good test of the theories given above and of the forecasting methods which were verified against the height observations.

12. Summary

In summary, based on the above results, the following rules can be given for the visual observations of wave heights:

(1) The heights of the waves passing a fixed point should be observed. (The point could also be fixed relative to a moving ship.)

(2) All heights should be recorded (or if this is too difficult, all heights in excess of a fixed lower bound should be observed and the theory of the truncated distribution then used).

(3) At least fifty values, preferably one hundred values, should be recorded.

(4) Table 3 then gives values for the confidence limits to be placed on the observations. The value is more exact theoretically if all waves are observed, and it is approximately correct when a truncated distribution is used.

III. VISUAL "PERIOD", "WAVE LENGTH", AND "SPEED" OBSERVATIONS

A. Definition of Terms

Part of the difficulty in making wave observations and wave record analyses lies in the loose interchange between theory and practice of two distinctly different meanings of the word, "period".

A period of a simple harmonic progressive wave is a number with a precise mathematical meaning. A true period will be underlined in this paper, and it will be designated by the symbol, T.

The time interval between two successive characteristic points in a wave record, such as the wave crests or the zero up-crosses, is not a period in the exact mathematical sence since a wave record is not periodic. These time intervals will be called "periods". A wave record has many different "periods". A simple sine wave has only one period. "Periods" in this sense will have quotation marks around them. The individual "periods" will be designated $\tilde{T}_j$ as they are enumerated in an observation or from a wave record; and the average "period", that is the average of all of the observed "periods", will be called $\tilde{T}$.

Similarly, wave length, (L), and "wave length", ($\tilde{L}$), will be discussed. For additional discussion of these terms, see Pierson (1954) and Pierson, Neumann, and James (1955).

Figure 5 illustrates the analysis of a wave record for its various

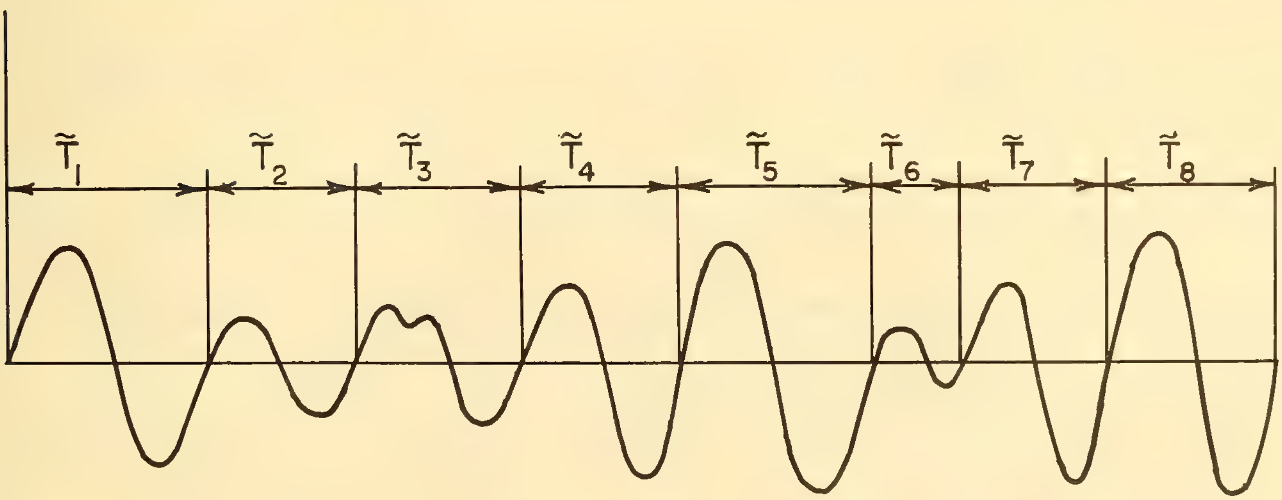

Figure 5. The Definition of the "Periods" in a Wave Record.

"periods". The "periods" are designated by T_1, T_2 and so on. In such a wave record it is theoretically wrong to equate these observed "periods" with the period of a simple harmonic progressive wave.

B. Visual "Period" Observations

1. Theory of the "Period" Distribution

The probability distribution function of the time intervals between successive wave crests is not known. Rice (1944) has given a formula that gives the mean of this unknown distribution in terms of an integral which involves the spectrum of the waves. Apparently none of the higher moments is known.

Even if the distribution of these "periods" were known, it would still tell us very little about the true spectral periods in the exact mathematical sense of the word. In addition, the loose interchange of "periods" and periods in theoretical work leads generally to invalid results.

2. Method of Observation

The observed statistical distribution of the "periods" and the average "period" for a given state of the sea are nevertheless useful values which can be obtained by the use of a stop watch in visual observations. Recommended procedures for observing the "periods" are given in Pierson, Neumann, and James (1955). A foam patch or a floating object can be used as a reference point. Two observers working as a team can make the observations more rapidly and efficiently.

C. Calculation Of The Average "Period" In Terms Of Theoretical Spectra

1. Method of Calculation

Neumann (1953) has shown good reason to believe that the spectrum of a sea grows from high frequency to low frequency with increasing duration or fetch. The spectrum of a given state of the sea is given by

$$\left[A(\mu) \right]^2 = \frac{C}{\mu^6} e^{-2g^2/v^2\mu^2} \quad \text{for } \mu > \mu_i \tag{24}$$

where μ_i is a function of the wind velocity and either the fetch or the duration, and v is the wind velocity.

The average "period" for a partially developed sea can be evaluated in terms of μ_i by the following procedure. Let $2\pi f = \mu$ and $2\pi f_i = \mu_i$, and let $f_i = 1/T_i$.
Then the use of a formula derived by Rice (1944) shows that the average "period" is given by

$$\tilde{\tilde{T}} = \left[\frac{\int_{f_i}^{\infty} \frac{1}{f^6} e^{-2a/f^2} df}{\int_{f_i}^{\infty} \frac{1}{f^4} e^{-2a/f^2} df} \right]^{1/2} \tag{25}$$

or by

$$\tilde{\tilde{T}} = \left[\frac{\int_0^{T_i} T^4 e^{-2aT^2} dT}{\int_0^{T_i} T^2 e^{-2aT^2} dT} \right]^{1/2}, \tag{26}$$

where

$$a = g^2/4\pi^2 v^2.$$

The integrals given in (26) can be integrated by parts until an integral involving the probability integral results. A change in variable under the final integral in order to put it into unit normal form yields, after several operations, the result that

$$\frac{\sqrt{3}\, 2\pi v}{2g} \left[1 - \frac{1}{\left[\frac{3e^{2g^2 T_i^2/4\pi^2 v^2}}{8T_i^3 g^3/(2\pi v)^3} \int_0^{2g T_i/2\pi v} e^{-\alpha^2/2} d\alpha - \frac{3}{4g^2T^2/(2\pi v)^2} \right]} \right]^{1/2} \tag{27}$$

The ratio, $gT_i/2\pi v$, occurs everywhere in equation (27). This is the ratio of the phase speed of the highest spectral period present to the wind speed. It is usually designated by

$$\beta_i = gT_i/2\pi v \tag{28}$$

and then equation (27) yields

$$\tilde{\tilde{T}} = \frac{\sqrt{3}\, 2\pi v}{2\ g} \left[1 - \frac{1}{\left[\frac{3e^{2\beta_i^2} \sqrt{2\pi}}{8\beta_i^3} \int_0^{2\beta_i} \frac{1}{\sqrt{2\pi}} e^{-\alpha^2/2} d\alpha - \frac{3}{4\beta_i^2} \right]} \right]^{1/2}. \tag{29}$$

As B_i approaches infinity, this equation reduces to

$$\tilde{\bar{T}} = \frac{\sqrt{3}}{2}\,\frac{2\pi v}{g} \tag{30}$$

in c.g.s. units or to

$$\tilde{\bar{T}} = 0.285v \tag{31}$$

where v is in knots. This is the average "period" of the fully developed sea as shown by Pierson (1954).

The function of B_i in equation (29) can be evaluated and used to determine T for a partially developed sea. Let the term in brackets in equation (29) be $F(B_i)$. Then equation (29) becomes

$$\tilde{\bar{T}} = 0.285v.F(\beta_i).$$

The values of $F(B_i)$ are given below in table 8.

Table 8

$F(B_i)$ as a function of B_i.

B_i	$F(B_i)$	B_i	$F(B_i)$
0.1	0.09	0.8	0.67
0.2	0.18	0.9	0.72
0.3	0.27	1.0	0.78
0.4	0.35	1.2	0.87
0.5	0.43	1.4	0.93
0.6	0.51	1.6	0.97
0.7	0.59	2.0	1.00

From the duration or fetch curves on the co-cumulative spectra given by Pierson, Neumann, and James (1955), the values of f_i and v for a particular weather situation can be determined. The use of table 8 above then permits a computation of the average "period" of the waves.

If B_i is low (less than 0.5), the value of T can be approximated from equation (26) in a more direct way. The exponential term under the integral can be expanded in series with the result that

$$\tilde{\bar{T}} = \left[\frac{\int_0^{T_i} \sum_{n=0}^{\infty} \frac{(-1)^n (2a)^n}{n!} T^{2n+4}\, dT}{\int_0^{T_i} \sum_{n=0}^{\infty} \frac{(-1)^n (2a)^n}{n!} T^{2n+2}\, dT}\right]^{1/2} \tag{33}$$

and that

$$\tilde{\bar{T}} = T_i \left[\frac{\sum_{n=0}^{\infty} \frac{(-1)^n \; (2a)^n}{n!} \; \frac{T_i^{\;2n}}{2n+5}}{\sum_{n=0}^{\infty} \frac{(-1)^n \; (2a)^n}{n!} \; \frac{T_i^{\;2n}}{2n+3}} \right]^{1/2} \tag{34}$$

The first few terms of this expansion are

$$\tilde{\bar{T}} = T_i \left[\frac{\frac{1}{5} - \frac{2}{7} \beta_i^2 + \frac{2}{9} \beta_i^4 - \cdots}{\frac{1}{3} - \frac{2}{5} \beta_i^2 + \frac{2}{7} \beta_i^4 - \cdots} \right]^{1/2} \tag{35}$$

and under the condition that B_i be small, a reasonable approximation is that

$$\tilde{\bar{T}} = T_i \sqrt{3/5} = 0.77\ T_i \tag{36}$$

as given by Pierson, Neumann, and James (1955).

2. Interpretation of the Average "Period"

The average "period" as observed by stop watch, or as computed from a wave record, can be an extremely misleading statistic. It overemphasizes the short "periods" and neglects the long "period". The maximum energy in the spectrum is always at a higher value than is indicated by the average "period".

The significant "period", that is, the average period of the one-third highest waves, may equal the average "period" or it may be a trifle higher because of the neglect of shorter "periods" in the average. However, it is even more doubtful a statistic because its relation to the wave energy spectrum is not known.

For either the average "period" or the significant "period", the computation of the average wave crest "speed" or the average "wave length" cannot be carried out by the use of the classical formulas as will be shown later. The classical formulas apply only to the true period of a simple harmonic progressive wave.

The average "period" can be used to determine the state of development of the sea for a given wind velocity. It can be used to check a given forecast of the wave spectrum if only a sea is present. However, spectra of many different shapes can yield the same average "period"; and the average "period" and the significant height do not completely characterize a given state of the sea.

In fact, there appears to be no way to obtain parameters which completely describe the seaway by visual observations or by the statistical analysis of a wave record or a pressure record. The partial characterization in terms of significant height and average "period" is, however, useful in many aspects if it is interpreted with care in terms of possible wave spectra and the meteorological synoptic situation.

D. "Wave Lengths"

1. The Observation of the "Wave Length"

Photographs of the sea surface, such as figure 1, show that it is composed of short-crested waves. There are medium waves superimposed on the big waves and short waves superimposed on the medium waves. There are ripples on top of everything else. The waves in a photograph are much more irregular than a corresponding wave record. There appear to be more short waves in a photograph than there are in a wave record.

Most of the time a dominant direction of travel can be determined for the waves. Then the length of the waves along this direction can be measured. The actual distance between successive crests must be measured. Procedures for measuring the "wave length" are given in Pierson, Neumann, and James (1955). The procedures involve towing a line with floats behind a vessel for use as a scale, and the use of the ship or other ships as a scale factor.

The average "wave length" cannot be computed from the average "period" in terms of the classical formula. Stated another way, it is not true that the average "wave length" in feet equals 5.12 times the square of the average "period" in seconds. For fully developed seas, the average "wave length", if the theoretical spectrum which is assumed is correct, is given by

$$\tilde{\bar{L}} = \frac{2}{3} \cdot 5.12\, \tilde{\bar{T}}^2 \qquad (37)$$

For the theory of the derivation, see Pierson (1954). For nonfully developed seas the formula does not hold, and the derivation of the average "wave length" is more difficult.

It appears that the p.d.f. of the "wave lengths" cannot be computed from the p.d.f. of the "periods" even if the p.d.f. of the "periods" were known. It would have to be computed by mapping the wave spectrum as a function of frequency and direction, into a frequency spectrum of the spectral wave lengths. Then, if the theory of the p.d.f. of the

"periods" is ever solved, it will be possible to determine the p.d.f. of the "wave lengths" by using the same theory on the new spectrum. Aerial photographs, if the scale is known, would be useful in the determination of the probability distribution of the "wave lengths" empirically.

Observations which show that the formula given above is more nearly correct for the average "wave length" than is the classical formula are cited by Dearduff (1953). He states that "the observed wave lengths were as a whole much smaller than the calculated lengths based on the usual formula." The value which was obtained from the analysis of observations made from Nantucket Lightship was half of the value which would be obtained using the classical formula.

The theory on which equation (37) is based assumes that ripples on top of the more dominant cycles are not counted in the measurement of the "wave lengths". The crest must be above sea level and the trough must be below sea level before the wave can be counted. A ripple or perturbation riding on top of a larger wave should not be counted. When such values are counted their effect is to decrease the average wave length to a value even less than the one given by equation (37).

2. Explanation of Theory of Equation (37)

There is an idea prevalent in current wave theory that a wave record can be broken up into pieces of one wave per cycle and that each oscillation can be treated as if it were a sine wave with the use of the classical formulas for the piece.

The theory can be sketched briefly as follows: Given a wave record as on the bottom of figure 6, the record is broken up into pieces at each zero up-cross and each fragment is treated as if it were a piece of a sine wave with a true period equal to the length in time of the piece and with an amplitude equal to one-half the crest-to-trough height of the piece. If the above assumptions were correct, then the wave record could be represented mathematically as the sum of a number of functions of the form sketched on the top of figure 6.

Such a representation is obviously absurd. If such a fragment were generated in a wave tank, it would alter in form completely before it could travel even a few feet. A Fourier analysis of one of the pieces shown in figure 6 would show it to be composed of a very broad Fourier spectrum of frequencies so that it would not be correct to apply the "period" $\tilde{T}_i$ to one of the pieces. Such a small piece of a sine wave is not the same thing as a sine wave.

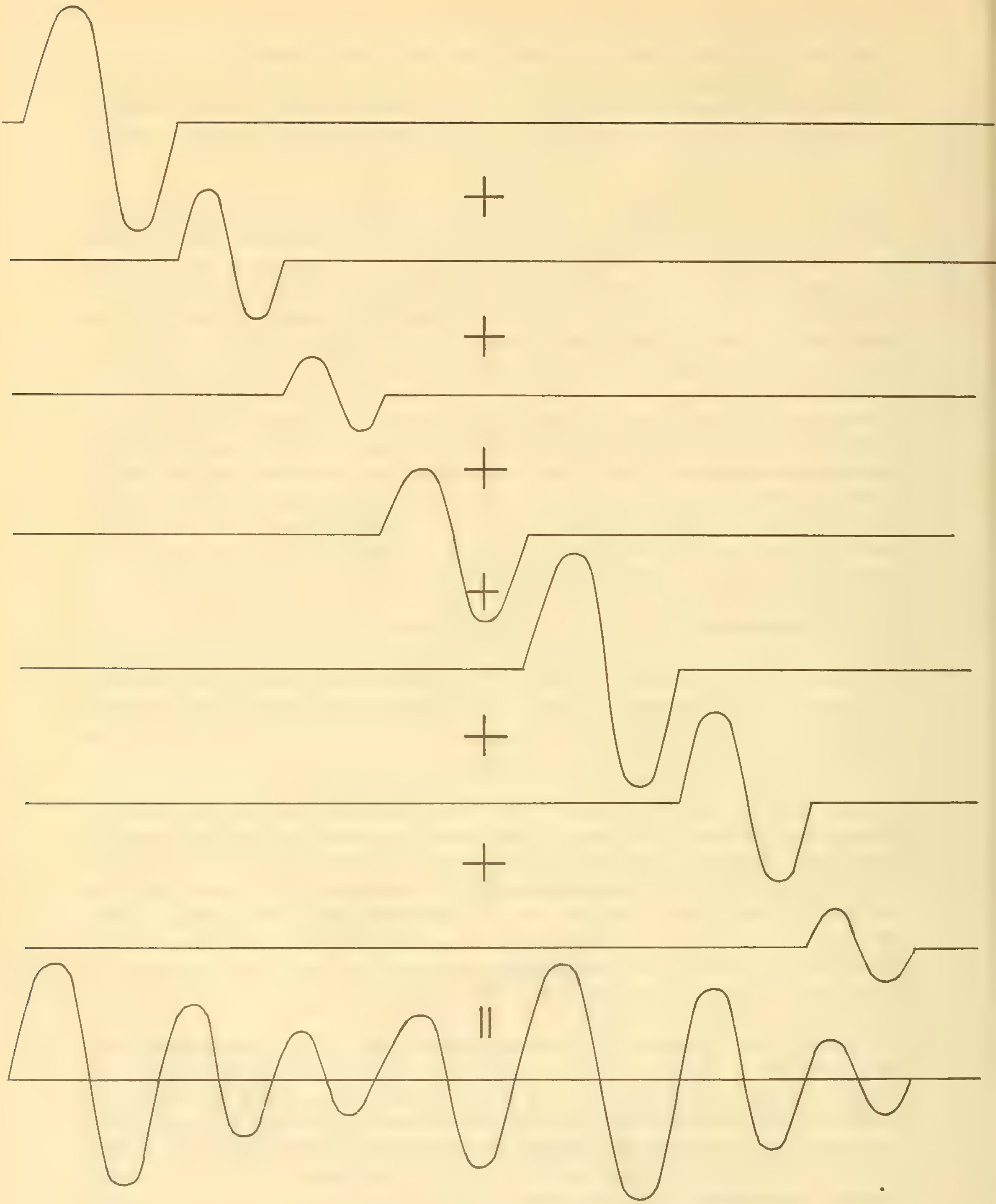

Figure 6. The Representation of a Wave Record as a Sum of Individual Sinusoidal "Cycles" with Different "Periods" in an Artificial Wave by Wave Analysis.

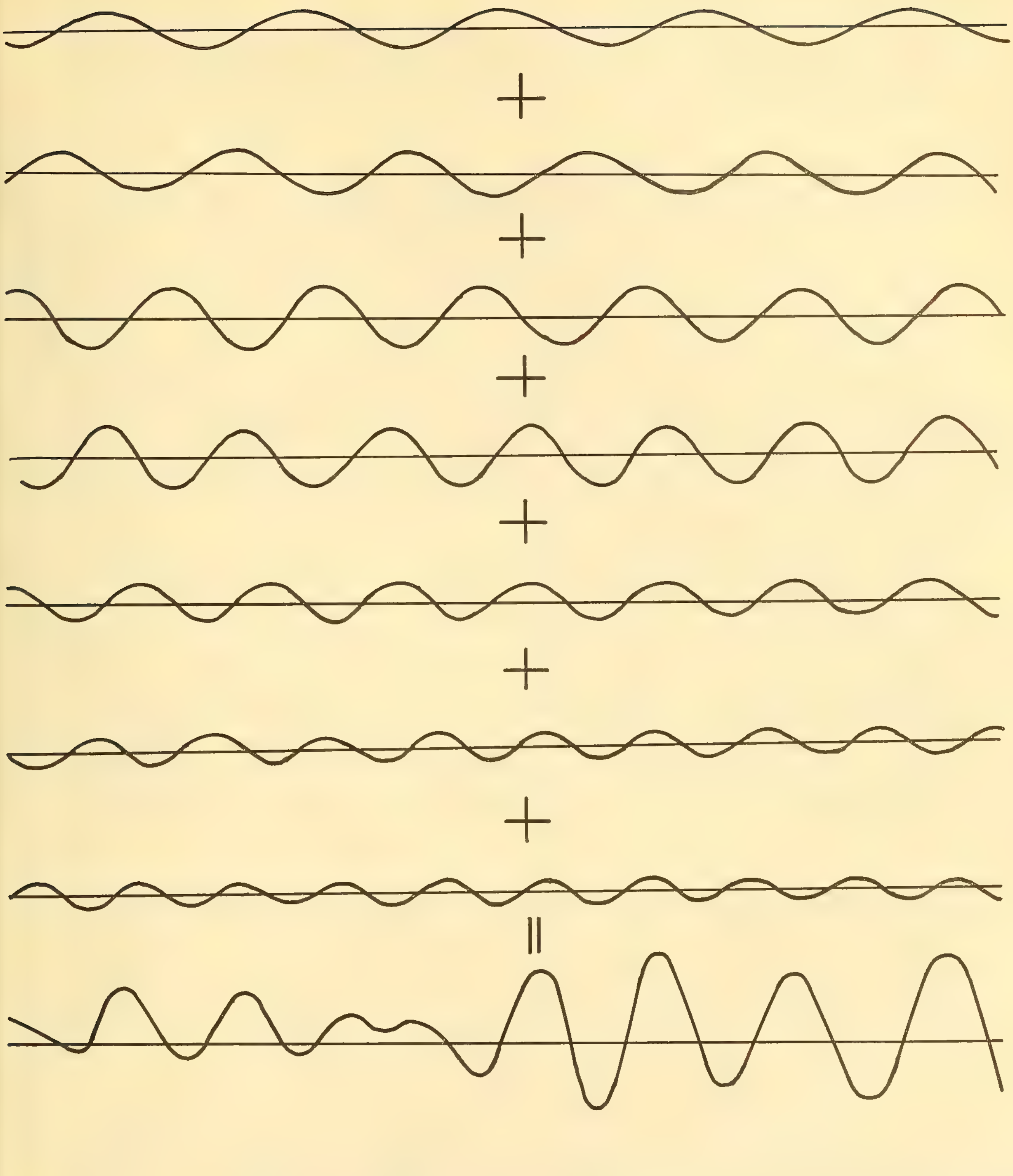

Figure 7. The Representation of a Wave Record as a Sum of Many Sine Waves with Individual True Periods.

The correct way to think of a wave record is to think of it as composed of a very large number of very low sine waves with phases all mixed up and with different periods in such a way that figure 7 represents the wave record.

If any one of the above pieces is generated for a long enough time in a wave tank, the waves propagate without change of shape and have classical <u>wave lengths</u> and classical phase speeds after initial transients have died out. Since, as a first approximation, the system combines linearly when all waves are produced simultaneously, the behavior of the sum equals the sum of the behaviors of the individual sinusoidal components.

Figure 7 explains why it is so difficult to observe the visual properties of waves or to analyze a wave record statistically. The variation in wave amplitudes described at the start of this paper is caused by the complicated effects of phase reinforcement and cancellation of this large (infinite) sum of small (infinitesimal) amplitude true sine waves combined in random phase.

It also explains the difficulties involved in determining the "periods", since a "period" is the time interval between two successive zero up-crosses. When a sum of, say, fifty or sixty true sine waves is written out and when they are assigned amplitudes according to some spectral law and phases at random, it then becomes difficult, if not impossible, to solve for those times in the record produced where the record adds up to zero and to compute the time intervals between the zeros. These "periods" thus are produced by an interference effect. This is why the probability distribution function of the "periods" is not known theoretically. Mathematicians simply have not yet been able to solve this problem.

Intuitively, at least, the reason why the average "wave length" is given by equation (37) in a fully developed sea can now be explained. If the wave crests were infinitely long, then corresponding to each sine wave in the sum as observed as a function of time at a fixed point, there would be a sine wave on the sea surface as a function of distance along a line.

Each <u>wave length</u> in feet would be given by 5.12 times the square of the true <u>period</u> of the sine waves in the sum which goes to make up the sea surface along the line. The <u>wave lengths</u> are related to the square of the <u>periods</u>. The more rapid oscillations in the record as a function of distance for <u>periods</u> less than the average "period" outweigh the effect of the much less rapid oscillations for <u>periods</u> greater than the

average "period", and the result is that the average "wave length" is less than what would be computed from the average "period" by the use of the classical formula. When this effect is corrected for the short-crestedness of the waves, the result is equation (37) for a fully developed sea.

As part of the erroneous method of wave analysis depicted in figure 6, it is frequently assumed that the "wave length" of the wave which passed during the time interval, $\tilde{T}_j$, is given by

$$\tilde{L}_j = 5.12\ \tilde{T}_j^2 \tag{38}$$

in deep water or by an appropriately modified equation in shallow water. This assumption is obviously dependent upon the assumption that the zero which passes at the start of the "cycle" does not disappear before the zero which passes at the close of the "cycle" finally arrives, and upon the assumption that a new zero does not form between the first zero and the point of observation and the old zero before the second recorded zero passes. (Similar remarks could be made about crests.) Since the wave forms of actual ocean waves do not propagate without change of shape, and since the crests of actual ocean waves are not conservative, these assumptions are not valid and the formula cannot be used.

The average of the "wave lengths" as computed from the individual "periods" is always greater than the average "wave length" computed from the average "period", and even this latter value is too big.

Although it is unknown, suppose that the p.d.f. has the typical properties of all p.d.f.'s in that it gives the probability that a "period" within a band of "periods" will be observed.

The p.d.f. of the "periods" is then g (T)dT with the properties that

$$g(\tilde{T}) = 0 \text{ for } \tilde{T} < 0, \tag{39}$$

$$g(\tilde{T}) \geq 0 \text{ for } \tilde{T} > 0, \tag{40}$$

and

$$\int_0^\infty g(\tilde{T})\, d\tilde{T} = 1 \tag{41}$$

The average "period" then equals

$$\bar{\tilde{T}} = \int_0^\infty \tilde{T} g(\tilde{T})\, d\tilde{T}. \tag{42}$$

The average "period" is estimated from a finite series of individual "period" observations such that

$$\tilde{\bar{T}} = 1/n \sum_{i=1}^{n} \tilde{T}_i \, . \tag{43}$$

Now consider the following form which is always greater than or equal to zero because it is the integral of an always positive (or zero) function.

$$\int_0^\infty (\tilde{T} - \tilde{\bar{T}})^2 \, g(\tilde{T}) \, d\tilde{T} \geq 0 \tag{44}$$

It yields

$$\int_0^\infty \tilde{T}^2 \, g(\tilde{T}) \, d\tilde{T} - 2\tilde{\bar{T}} \int_0^\infty \tilde{T} g(\tilde{T}) \, d\tilde{T} + \tilde{\bar{T}}^2 \int_0^\infty g(\tilde{T}) \, d\tilde{T} \geq 0 \tag{45}$$

or

$$T^{*2} = \int_0^\infty \tilde{T}^2 \, g(\tilde{T}) \, d\tilde{T} \geq \tilde{\bar{T}}^2 \tag{46}$$

The term on the left is estimated by

$$T^{*2} = \frac{1}{n} \sum_{i=1}^{n} \tilde{T}_i^2 \tag{47}$$

Now let L* be the average "wave length" computed by computing the "wave length" associated with each of the observed "periods" and averaging the results. From equation (47), this "wave length" is given by

$$L^* = \frac{g T^{*2}}{2\pi} = \frac{1}{n} \sum_{i=1}^{n} \frac{g \tilde{T}_i^2}{2\pi} \tag{48}$$

The wave length, L_A, computed from the average "period" is found by averaging the observed "periods" and computing the average "wave length" from the average "period" according to equation (43).

$$L_A = \frac{g \tilde{\bar{T}}^2}{2\pi} = \frac{g}{2\pi} \left(\frac{1}{n} \sum_{i=1}^{n} \tilde{T}_i \right)^2 \tag{49}$$

But from equation (46), L* is greater than L_A, and from equation

(37) L_A is too big when compared with actual observations. Therefore the average "wave length" cannot be computed from the observed distribution of the "periods". The individual "wave lengths" computed from the individual "periods" have therefore a very doubtful meaning.

With reference to "wave lengths", the only reliable formula is for the average "wave length" for a fully developed sea as given by equation (37). For swell, the average "wave length" is approximately given by the classical formula using the average "period" of the swell. For seas not fully developed or for cross seas, no convenient formulas, in general, exist.

However, for newly generated partially developed seas in which B_i is less than 0.5, it is possible to obtain an approximate value for $\tilde{\bar{L}}$. Under these conditions, $\tilde{\bar{L}}$ is given by

$$\tilde{\bar{L}} = 2.56 \; T_i^{\,2} \tag{50}$$

The method for deriving equation (50) involves short-crested seas and employs approximations and procedures similar to those used in equations (33) through (36).

E. Wave "Speeds"

1. Theory - A Contradiction

The usual wave observation procedure has been that of observing the "periods" of the waves and computing the average "period". The "wave lengths" and "speeds" of the individual waves are rarely independently observed.

The theories given above suggest that the average "wave length" of a fully developed sea is two-thirds of the value given by the classical formula. Also some independent observations suggest that these theories are more nearly correct.

In c.g.s. units, the two classical formulas for the speed of a wave crest are given by

$$C = L/T \tag{51}$$

and

$$C = gT/2\pi \;. \tag{52}$$

In terms of average "periods" and average "wave lengths" in

units, equation (37) becomes

$$\tilde{\tilde{L}} = (\tfrac{2}{3})\, g\tilde{\tilde{T}}^2/2\pi \tag{53}$$

Now suppose that the average speed is computed by assuming that the classical formulas involving the period and the wave length of a simple harmonic progressive wave hold for the average "period" and the average "wave length" of an irregular state of the sea. The results are that

$$\tilde{\tilde{C}} = \frac{2}{3}\, g\tilde{\tilde{T}}/2\pi \tag{54}$$

from equations (51) and (53) and that

$$C = gT/2\pi \tag{55}$$

from equation (52).

The result is two different values for the same theoretical quantity, and there is a contradiction involved. The contradiction lies in the assumption that the classical formulas can be applied to average wave properties.

For an irregular sea, current theory tells us nothing about the average wave "speed". Neither equation (54) nor (55) can be assumed to be the correct one.

2. The Observation of Wave "Speeds"

Wave crest "speeds" must therefore be observed independently of the "periods" and the "wave lengths". The "speed" of a given crest may not even be a constant. The wave crest "speeds" can be measured at the same time that the "wave lengths" are being measured by the methods given by Pierson, Neumann, and James (1955). Such observations in a sea are very scarce, if any exist at all, and thus the present state of theory and observation can give no information on this problem. Data on this problem, when they become available, will prove to be very interesting.

IV. CONCLUSIONS

The visual observation of the properties of ocean waves will always be an important supplementary source of wave data. The data thus obtained can never be as adequate as wave records which are analyzed for their spectra, but they can be used if they are interpreted with care.

A series of wave height observations can be used to verify the theoretical probability distribution of wave heights. Even for irregular seas, the distribution may be fairly well approximated. The reliability of the average height can be estimated from the size of the sample and confidence limits can be assigned to the values observed. The theory of truncated distributions can be used to refine the values if the low waves are neglected.

The average "period" is a misleading statistic unless it is interpreted in terms of the wave spectrum. It gives a value which is shorter than the period where the maximum energy exists in the spectrum. It can be forecast and thus related to the spectrum of the waves.

The average "wave length" cannot be computed with the use of the average "period" by means of the classical formulas. For a fully developed sea in deep water the theoretical value is two-thirds of the value that results from the classical theory. The "wave length" of an individual wave cannot be computed from the "period" of that wave as it passes a fixed point.

The wave crest "speeds" are rarely observed, and the classical formulas cannot be used to predict the "speeds" from the "periods" and "lengths" in a sea.

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ADDENDUM

Since the preparation of this paper and of the wave forecasting manual (H.O. Pub. No. 603), two papers of interest in connection with this paper have come to the attention of the author. The first paper, by Arakawa and Suda (1953), gives some data on the measurement of the average wave length of a wind-driven sea. The second paper, by Wooding, gives information on the meaning of the significant period.

Arakawa and Suda (1953) summarize and discuss some wave observations made by the Japanese Navy during a typhoon which occurred on September 26, 1935. Table 6 of their paper is reproduced in full and a paragraph referring to the table is quoted as follows:

"Table 6 shows that comparisons of measured and computed values for the MIKUMA gave rather unsatisfactory results. This may indicate that the state of the sea as observed by the main squadron was, to some extent, uncertain. Comparisons of measured and computed values for the wave length and the wave period from the cruiser NACHI on the other hand gave fairly satisfactory results."

It should be noted that according to the notation used in this text, $\tilde{\tilde{L}}$, $\tilde{\tilde{T}}$, and $\tilde{\tilde{C}}$ are probably what were really observed.

Table 6. Observed and computed values of velocities, lengths, and periods of wind waves in the typhoon area, Sept. 26, 1935

Name of ship	Wave velocity C, m. sec.			Wave length L,m.			Wave period T, sec.		
		computed from			computed from			computed from	
	Observed	$1.25\sqrt{L}$	1.56 T	Observed	$0.641C^2$	$1.56T^2$	Observed	$0.801\sqrt{L}$	0.641C
Mikuma 1250 JMT	14.3	16.8	20.3	180	131	264	13.0	10.7	9.2
Mikuma 1445 JMT	14.3	17.7	20.7	200	131	276	13.3	11.3	9.2
Nachi 1500 JMT	9	13.7	14.0	120	52	126	9	8.8	5.8
Nachi 1550 JMT	About 8	13.7	14.0	About 120	41	126	About 9	8.8	5.1

For a simple sine wave, the formulas, $L = CT$, and $L = gT^2/2\pi$ imply also that $L = 2\pi C^2/g$, $C = \sqrt{gL/2\pi}$, $C = gT/2\pi$, $T = \sqrt{2\pi L/g}$, and $T = \sqrt{2\pi C/g}$. Thus if L or C or T is observed, the other two quantities can be computed from it. In table 6, the comparison shows that C could not be predicted from either T or L. The value of L when computed from C is much too low. When L is computed from T the NACHI observations agree, but the computed wave length is considerably greater than the observed wave length in the MIKUMA observations. When the formula for the average wave length $\tilde{\bar{L}}$, in terms of the average period, $\tilde{\bar{T}}$, namely $\tilde{\bar{L}} = \frac{2}{3} g \tilde{T}^2/2\pi$, is applied to the MIKUMA observations, the period of 13 seconds yields a value of 2/3 of 264 meters or 176 meters as compared to an observed wave length of 180 meters. The second set of observations yields a value of 184 meters as compared to an observed value of 200 meters. The percentage error with respect to the observed average wave length is about 2% with the new formula and 47% with the classical formula in the first case. In the second case, the errors are 8% and 38 %, respectively.

It is most interesting that two of these four sets of observations obtained in 1935 should agree with the newly derived formula. Since the other two do not, it can be added that observations in a towing tank in which Gaussian waves were generated, confirm the theoretical basis of the derivation of the new formula.*

Wooding (1955) has derived an approximate joint probability distribution for wave amplitude and frequency (period) in random noise, and he has applied the results to the interpretation of wave observations. The results show that the time interval between the successive upcrosses in a wave record has a higher probabiliy of being large if the wave is high than if the wave is low. Thus the average time interval between successive crests of the one-third highest waves should be greater than the average time interval between all the crests. Or, stated another way, the significant "period" is greater than the average "period."

It should be possible to derive a formula for the significant "period" in terms of a theoretical wave spectrum using the results of Wooding (1955). If an average wave length were obtained using the "significant" period and the classical formula, the error would be even greater than that obtained by using the classical formula and the "average" period.

In view of the difficulty of observing the significant wave height discussed in this paper, it is believed that the observation of a true

* See Lewis (1954).

significant "period" would be even more difficult, and that the conclusions of this paper with respect to visual wave observations should still be adhered to substantially. The rules given are internally consistent, and should yield consistent results.

U. S. Navy Hydrographic Office
VISUAL WAVE OBSERVATIONS, by Willard J. Pierson, Jr., Dept. of Met. and Oc., New York U., March 1956. 50 p., including 7 figures, 8 tables. (H. O. Misc. 15921).
Bibliography: 46-47
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